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2000 J. Phys. A: Math. Gen. 33 2031

(http://iopscience.iop.org/0305-4470/33/10/306)

J. Phys. A: Math. Gen. **33** (2000) 2031–2044. Printed in the UK PII: S0305-4470(00)08853-3

The high-temperature expansion for the Potts model and nowhere-zero flows

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Received 20 October 1999

Abstract. It is pointed out that a graph contributes to the high-temperature expansion of the M-state Potts model if and only if it has a nowhere-zero M-flow. This graph-theoretical property is shown to be equivalent to the existence of a proper arrowing of the graph, which can be visualized easily by means of arrows put on the edges in such a way that the sum of these arrows is 0 mod M at every vertex. By means of this concept it is rather easy to solve the problem for planar graphs completely, one result being that all planar graphs contribute for $M \geqslant 4$. For nonplanar graphs, all of them certainly contribute for $M \geqslant 6$, but it is an unproven conjecture that $M \geqslant 5$ suffices. The classes of graphs contributing for $2 \leqslant M \leqslant 6$ are characterized.

1. Introduction

The partition function for the M-state Potts model on a lattice graph L can be written as

$$Z(L) = \sum_{\substack{\{i_v\} \\ v \in V(L)}} \prod_{e \in E(L)} \Omega(i_{v_1(e)}, i_{v_2(e)}) \tag{1.1}$$

where $V(L)$ is the set of vertices of the lattice, $E(L)$ the set of its edges connecting interacting spins, each vertex variable takes on M different values, $i_v \in \{0, 1, \ldots, M-1\}$, and the vertices at the ends of edge $e \in E(L)$ are denoted by $v_1(e)$ and $v_2(e)$. The Boltzmann factor matrix occurring for each pair of interacting spins is given by

$$\Omega(i, j) = \begin{cases} 1 & \text{for} \quad i = j \\ \omega = \exp(-E/k_{\mathrm{B}}T) & \text{otherwise.} \end{cases} \tag{1.2}$$

Here E is the mutual energy of two spins in different states; the energy for the case of equal states has been set to 0. From this, a high-temperature expansion [1–3] can be derived by rewriting the Boltzmann factors as

$$\Omega(i, j) = [1 + (M-1)\omega][1 + q(i, j)]/M. \tag{1.3}$$

The matrix $q(i, j)$ measures the deviation from the infinite-temperature limit:

$$q(i, j) = \tilde{\omega} Q(i, j) \qquad \tilde{\omega} = (1-\omega)/[1 + (M-1)\omega]$$
$$Q(i, j) = \begin{cases} M-1 & \text{for} \quad i = j \\ -1 & \text{otherwise.} \end{cases} \tag{1.4}$$

It is clear that the dual Boltzmann factor $\tilde{\omega}$ is the small parameter for the expansion. Insertion of equations (1.4) into the partition function of eqaution (1.1) gives an expansion in terms of powers of $\tilde{\omega}$; the coefficients in this power series are sums of products $\alpha(G, M)$ for multiply connected subgraphs of L. These subgraphs are defined by the edges carrying a q in the explicit multiplication of the $[1 + q(i, j)]$-factors:

$$\alpha(G, M) = M^{-|V(G)|} \sum_{\substack{\{i_v\} \\ v \in V(G)}} \prod_{e \in E(G)} Q(i_{v_1(e)}, i_{v_2(e)}). \tag{1.5}$$

Here the notation $|A|$ is used for the cardinality of the set A. For more details concerning this expansion and the corresponding one for the free energy, see, e.g., [1, 3, 4].

The question to be treated in this paper is: for which graphs and for what number of states does equation (1.5) give a nonzero result? This is motivated by the well known fact [1, 2] that for the case of the Ising model ($M = 2$), only graphs with an even number of edges at each vertex can give a contribution. Also, although equation (1.5) is only defined for subgraphs of a lattice graph, it is easy to see that every possible multiply connected graph will make a contribution:

- If the lattice graph is planar (i.e., the Potts model is considered in two dimensions), an arbitrary planar graph G_p can always be approximated: G_p can be embedded in the plane so that there is a minimal distance d between any pair of nonconnected edges, i.e., edges without a common vertex; taking the lattice constant much less than d, the graph can be approximated by edges of the lattice graph without unwanted intersections. Since equation (1.4) implies that
$$\mathsf{Q}^2 = M\mathsf{Q} \tag{1.6}$$
holds, equation (1.5) guarantees that $\alpha(G_p, M)$ is identical to this function for the approximating lattice subgraph.
- If the Potts model is considered to be defined on a three- or higher-dimensional lattice, the situation is similar: since *every* graph G can be embedded into three-dimensional space [5] so that all pairs of nonconnected edges have a minimal distance d again, a sufficiently small lattice constant can give rise to a lattice subgraph with the same $\alpha(G, M)$.

This paper is organized as follows. In section 2, a number of cases in which $\alpha(G, M)$ factorizes is studied to provide restrictions on the types of graphs which have to be taken into account. In section 3 the connection of the present problem with the Tutte polynomial, nowhere-zero M-flows and proper arrowing is discussed. Also, the concepts of Ising thickness (the minimal number of even-valency (Ising) graphs covering a graph), and of arrow number (the minimal value of M, for which the graph contributes) are shown to be well defined. Section 4 is devoted to a rather detailed description of the reduction of these problems to the corresponding ones for cubic or trivalent graphs. With these preliminaries, the problem is solved for graphs with Ising thickness 2 in section 5, which case includes the planar graphs. In section 6, finally, the graphs occurring in the general (dimension $\geqslant$3) case are classified and a short discussion about the practical determination of the arrow number is given.

2. Restrictions on the graphs

In this section, it will be shown, that equations (1.4) and (1.5) restrict the types of graphs that have to be considered rather severely. First of all, the graphs can be taken such that they have no loops (edges starting and ending at the same vertex), since such loops simply give an extra factor $(M - 1)$ by equation (1.4). This is a special case of a graph containing articulation points (i.e. vertices, upon removal of which the graph becomes disconnected) and it is easy to see that the transitivity of the Potts model group $\mathcal{S}(M)$ leads to a factorization

$$G = G_1 \cup G_2 \qquad G_1 \cap G_2 = \{v\} \Rightarrow \alpha(G, M) = \alpha(G_1, M)\alpha(G_2, M). \tag{2.1}$$

Also, the graphs contain no isthmuses, i.e., edges which upon cutting them disconnect the graph, since equations (2.1) can then be used twice to give a product containing a factor $\alpha(e, M)$, which is zero by the definition of equation (1.4), see also equation (3.4) below. In graph theory, one says that a graph is k-edge connected if cutting k edges (this is called a k-cut) cannot disconnect it. Similarly, a graph is k-connected if the removal of k vertices does not disconnect it. From the above, the graphs that have to be taken into account are at least 2-edge-connected and 2-connected.

There are several more cases, in which the function $\alpha(G, M)$ factorizes. The first of these occurs if G is exactly 2-edge-connected, i.e., if there are edges e_1 and e_2 so that cutting these disconnects the graph. Let these disconnected pieces be G_1, containing the endvertices a of e_1 and b of e_2, and G_2, containing the other end-vertices, c and d. Then one has

$$\alpha(G, M) = \sum_{i_a, i_b, i_c, i_d} A_1(i_a, i_b) A_2(i_c, i_d) Q(i_a, i_c) Q(i_c, i_d) \tag{2.2}$$

where the matrices A_1 and A_2 are obtained by summing over all internal vertex variables of G_1 and G_2, respectively. By the Potts symmetry $\mathcal{S}(M)$, these matrices necessarily have the form

$$A(i, j) = \begin{cases} B & \text{for all} \quad i = j \\ C & \text{for all} \quad i \neq j. \end{cases} \tag{2.3}$$

Now equation (2.2) is easily evaluated as a trace; comparing this with the values of $\alpha(H_1, M)$ and $\alpha(H_2, M)$, where the H_s are obtained from the G_s by connecting their vertex pairs a, b and c, d by new edges, one sees that the product formula

$$\alpha(G, M) = \alpha(H_1, M)\alpha(H_2, M)/(M - 1) \tag{2.4}$$

holds. Similarly, let G be exactly 3-edge-connected, i.e. the cutting of the three edges e_1, e_2 and e_3 disconnects the graph, so that there are disjoint subgraphs G_1 and G_2 with three external vertices each. Summing over all internal vertex variables now gives three-variable functions $A_s(i, j, k)$, which, by the Potts symmetry again, must be of the form

$$A(i, j, k) = \begin{cases} B & \text{if} \quad i = j = k \\ C_1 & \text{if} \quad i = j \neq k \\ C_2 & \text{if} \quad i = k \neq j \\ C_3 & \text{if} \quad j = k \neq i \\ D & \text{if all} \quad i, j, k \text{ are pairwise different.} \end{cases} \tag{2.5}$$

Let for the present case the graphs K_s be defined by taking a new vertex and connecting it by three edges to the external vertices of the G_s; the structure of equation (2.5) is simpler for this special case, since one has

$$\sum_{\alpha} Q(i, \alpha) Q(j, \alpha) Q(k, \alpha)$$
$$= \begin{cases} M(M-1)(M-2) & \text{for} \quad i = j = k \\ -M(M-2) & \text{for} \quad i = j \neq k \quad i = k \neq j \quad j = k \neq i \\ 2M & \text{for} \quad i \neq j \neq k \quad i \neq k. \end{cases} \tag{2.6}$$

A certain amount of algebra shows that the factorization

$$\alpha(G, M) = \alpha(K_1, M)\alpha(K_2, M)/[(M - 1)(M - 2)] \tag{2.7}$$

holds nonetheless for $M > 2$. This shows that it suffices to consider graphs which are at least 3-edge-connected, such that all 3-cuts are trivial, i.e. these disconnect, at most, a vertex of valency 3.

3. Nowhere-zero flows and proper arrowings

If G is any graph and e one of its edges, use of the defining relation $Q(i, j) = M\delta(i, j) - 1$ for this particular edge yields

$$\alpha(G, M) = \alpha(G/e, M) - \alpha(G - e, M) \tag{3.1}$$

where G/e is derived from G by contracting the edge (thus identifying its pair of end-vertices), whereas $G - e$ is obtained by deleting it (and using equation (1.6) twice). Now there is a unique polynomial function $T_G(x, y)$, the Tutte polynomial [6–8], defined on all graphs, with the properties:

(i) $T_{E(m)}(x, y) = 1$ for all edgeless graphs $E(m)$ with m vertices and
(ii) for any other graph, it can be calculated recursively by the relations

$$T_G(x, y) = \begin{cases} xT_{G-e}(x, y) & \text{if} \quad e \text{ is an isthmus} \\ yT_{G-e}(x, y) & \text{if} \quad e \text{ is a loop} \\ T_{G-e}(x, y) + T_{G/e}(x, y) & \text{otherwise.} \end{cases} \tag{3.2}$$

It is clear that there must be a relation; actually, one has [6, 7]

$$\alpha(G, M) = (-1)^{|V(G)|} T_G(0, 1 - M). \tag{3.3}$$

It is not surprising that this universal graph function turns up here, since it is well known that the partition function of the Potts model on a graph can also be expressed in terms of it [7, 8]. The same is true for the partition function of the more general random cluster model [7–9]. These recursion relations are not very effective for calculating these quantities, however.

It is possible to use equation (3.3) to derive the relation with nowhere-zero M-flows [6]. It is, however, easier to remark that equation (1.4) implies that the eigenvalues and eigenvectors of the matrix Q are known:

$$\sum_{l=0}^{M-1} Q(k, l) = 0 \qquad \text{for all} \quad k \tag{3.4}$$

so that the vector $\boldsymbol{e}$ with all entries equal to 1 is an eigenvector with eigenvalue 0. Now if $\boldsymbol{b}$ is any vector orthogonal to $\boldsymbol{e}$, which means that $\sum_l b(l) = 0$ holds, one has

$$\sum_{l=0}^{M-1} Q(k, l)b(l) = (M - 1)b(k) - \sum_{l \neq k} b(l) = Mb(k) \tag{3.5}$$

so that every such vector is an eigenvector with eigenvalue M. Further results depend on a choice of basis of this $(M - 1)$-dimensional space. Since the group of the Potts model is the full symmetric group $\mathcal{S}(M)$, any regular, Abelian subgroup of this will give such a basis by means of its dual group of characters: see, e.g., [10, ch 7]. In particular, if $\mathcal{C}(M)$ is the cyclic group on M objects, these characters are the Mth primitive roots of unity, which define a normalized basis of M-dimensional space by

$$b_m(l) = M^{-\frac{1}{2}} \exp(2\pi \mathrm{i} ml/M) \qquad m = 0, 1, \ldots, M - 1. \tag{3.6}$$

These are, for $m \neq 0$, orthogonal to $\boldsymbol{e}$, so that $Q(k, l)$ can be written as

$$Q(k, l) = \sum_{m=1}^{M-1} \exp[2\pi \mathrm{i} m(k - l)/M]. \tag{3.7}$$

In order to insert this into equation (1.5), the edges of G have first to be oriented, otherwise it is not clear to which vertex k or l belong; if the edge e is interpreted as pointing from v_2 to v_1, one obtains

$$\begin{aligned} \alpha(G, M) &= M^{-|V(G)|} \sum_{\substack{m_e=1 \\ e\in E(G)}}^{M-1} \prod_{v\in V(G)} \sum_{j_v=0}^{M-1} \exp[2\pi \mathrm{i}\, j_v K(v)/M] \\ K(v) &= \sum_{e\in E^+(v)} m_e - \sum_{e\in E^-(v)} m_e. \end{aligned} \tag{3.8}$$

Here the set $E^+(v)$ consists of all edges incident on v and pointing towards it, $E^-(v)$ of all edges pointing away from it. The sum over j_v gives zero, unless $K(v) = 0 \bmod M$ holds:

$$\alpha(G, M) = \sum_{\substack{m_e=1 \\ e\in E(G)}}^{M-1} \prod_{v\in V(G)} \delta_M[K(v)] \qquad \delta_M(x) = \begin{cases} 1 & \text{if} \quad x = 0 \bmod M \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

Clearly, $\alpha(G, M)$ is equal to the number of ways to label the edges by indices m_e taking values from $\{1, 2, \ldots, M-1\}$, so that, for a given orientation of these edges, all δ_M-restrictions are obeyed. Note that this number does not depend upon this edge orientation: reversing one arrow and replacing m_e by $M - m_e$ does not change the value of any $K(v)$ and the new variable has the same range as the original one. Such a labelling of the edges of a graph is called a *nowhere-zero M-flow*. Its independence from the particular orientation chosen shows that it is an intrinsic property of the underlying nondirected graph. As early as 1954, Tutte [6] conjectured that every graph without an isthmus has a nowhere-zero 5-flow. This result has not been proved yet. It is, however, known that every such graph has a nowhere-zero 6-flow [11], a result that was preceded by a proof of the existence of nowhere-zero 8-flow [12].

It is often more convenient to work with an *arrowing* instead of a nowhere-zero flow. This is defined for a specific configuration of edge variables and for a fixed edge orientation by:

- If the edge points from vertex 1 to vertex 2 and the edge variable is $m < M/2$, then put m arrows on this edge pointing from 1 to 2.
- If the edge points from 1 to 2 and the edge variable is $m > M/2$, then put $M - m$ arrows on this edge pointing from 2 to 1.
- If the edge points from 2 to 1 and $m < M/2$, there are m arrows from 2 to 1.
- If the edge points from 2 to 1 and $m > M/2$, there are $M - m$ arrows from 1 to 2.
- If $m = M/2$, put no arrows on this edge. Such edges, which can only exist for even M, will be called thick and drawn as such in figures.

Obviously, given an arrowing and the original edge orientation, the original edge variables can be retrieved unambiguously. The connection between the δ_M functions occurring in equation (3.9) and the restriction on the arrowing are easily found: let α^+ be the total number of arrows pointing towards a vertex v, α^- the number pointing away from it and α^0 the number of adjacent thick edges. Then these numbers have to satisfy the relation

$$\alpha^+ - \alpha^- + \alpha^0(M/2) = 0 \qquad \bmod M. \tag{3.10}$$

An arrowing such that equation (3.10) is fulfilled at every vertex is called a *proper M*-arrowing of the graph G. It is obviously the same as a particular nowhere-zero M-flow. If a graph has a nowhere-zero M-flow such a proper arrowing exists and the graph is called M-arrowable. As a direct consequence of these definitions, one can formulate the following, in which no explicit recourse to the group $\mathcal{C}(M)$ is necessary:

(a) A graph is $(M = 2k+1)$-arrowable if it is possible to orient every edge with $m \leqslant k$ arrows so that the number of ingoing arrows at every vertex equals the number of outgoing ones.

(b) A graph is $(M = 2k)$-arrowable if a set of thick edges exists so that the others can be oriented with $m < k$ arrows in such a way that equation (3.10) is satisfied at every vertex.

One can now interpret the number $\alpha(G, M)$ defined by equation (1.5) as the number of proper M-arrowings of G or, equivalently, as the number of nowhere-zero M-flows on G. A graph G gives a contribution to the high-temperature series for the M-state Potts model if and only if this graph is M-arrowable. This immediately implies the result for the Ising model cited above: for $M = 2$, there are only thick edges, counting 1 each in equation (3.10), so that a nonzero value for $\alpha(G, 2)$ can only be obtained if all vertices of G have even valency. Since this proper arrowing is unique, this nonzero value is actually equal to 1. A graph with vertices of even valency only contains an Eulerian circuit, i.e. a closed path starting and ending at an arbitrary vertex and traversing every edge of the graph exactly once, see, e.g., [13, p 159]. Now if all edges are given exactly one arrow in the direction of this Eulerian circuit, the resulting arrowing is proper for all M, since at every vertex of valency $2q$, there will be exactly q arrows pointing towards this vertex and just as many pointing away from it. This is a special case of a basic property of nowhere-zero flows [6, 7] stating that a graph with a proper M-arrowing also has a proper $(M + 1)$-arrowing, so that one can define an *arrow number* $\mu(G)$ for every graph as the smallest M for which a proper arrowing exists. The result of Seymour [11] implies that $\mu(G) \leqslant 6$ holds for every graph without an isthmus. For a graph with an isthmus, there are no nowhere-zero flows and one sets $\mu(G) = \infty$.

The above results have been derived by using as a basis the characters of the cyclic group $\mathcal{C}(M) \subseteq \mathcal{S}(M)$, which exists for all M. Equivalent results must be obtainable if M is such that other regular, Abelian subgroups of $\mathcal{S}(M)$ exist. Such an Abelian group $\mathcal{A}$ can always be written as the direct product of a number of cyclic ones [14]:

$$\mathcal{A} = \mathcal{C}(s_1) \otimes \mathcal{C}(s_2) \otimes \cdots \otimes \mathcal{C}(s_q) \qquad |\mathcal{A}| = M = s_1 s_2 \ldots s_q. \tag{3.11}$$

If a pair of the s_i have a common factor, this group is different from $\mathcal{C}(M)$. The basis vectors defined by such a group are simply direct products of the ones from equation (3.6):

$$b_{m_1,m_2,\ldots,m_q}(l_1, l_2, \ldots, l_q) = M^{-\frac{1}{2}} \prod_{j=1}^{q} \exp(2\pi \mathrm{i} m_j l_j / s_j). \tag{3.12}$$

In this basis, the matrix $Q(k, l)$ has the form

$$Q(k, l) = \sum_{m_1=0}^{s_1-1}{}' \ldots \sum_{m_q=0}^{s_q-1}{}' \prod_{j=1}^{q} \exp[2\pi \mathrm{i} m_j (k_j - l_j)/s_j] \tag{3.13}$$

where $k = k_1 + k_2 s_1 + k_3 s_1 s_2 + \cdots$ and l is given by an analogous expression. The primes on the sums indicate that $(m_1, m_2, \ldots, m_q) \neq (0, 0, \ldots, 0)$ must hold. Insertion of this form into equation (1.5) gives the following results:

(i) If G_j denotes the subgraph of G defined by the edges carrying a nonzero value of m_j, the summation over the jth component of the vertex spin gives the same type of expressions as for the simple case of equation (3.9), but with δ_{s_j}-functions. Therefore, this subgraph can be arrowed in the same way as before, but with numbers of arrows $\leqslant (s_j/2)$.

(ii) Since no edges can carry a zero for all j, the union of these G_j is G. It is important to remark that these partial graphs do not have to be different pairwise; also, they are not necessarily connected and can contain articulation points, but no isthmuses. This then shows, that the relation

$$\alpha(G, s_1 s_2 \ldots s_q) = \sum_{\substack{\{G_j\} \\ \cup_{j=1}^{q} G_j = G}} \prod_{j=1}^{q} \alpha(G_j, s_j) \tag{3.14}$$

must hold. This equation implies that a graph G is M-arrowable for $M = s_1 s_2 \ldots s_q$ if and only if it is the union of q graphs G_j, which are s_j-arrowable. In particular, a graph G is 2^q-arrowable if and only if it is the union of q graphs with even-valency vertices throughout. The value of $\alpha(G, 2^q)$ is exactly the number of ways in which such a decomposition is possible. Since these partial graphs do not have to be different, it follows immediately that a graph which is 2^q-arrowable is also 2^{q+a}-arrowable for all $a > 0$, so that the next definition makes sense: the *Ising thickness* $t(G)$ of a graph G is the minimal number of even-valency graphs covering G.

In view of Jaeger's theorem [12], every graph without an isthmus has Ising thickness $t(G) \leqslant 3$. The relation between Ising thickness and the arrow number is easily seen to be given by

$$t(G) = \lceil {}^2\log \mu(G) \rceil \qquad \text{or} \qquad 2^{t(G)-1} < \mu(G) \leqslant 2^{t(G)}. \tag{3.15}$$

Therefore, $t(G) = 1$ is equivalent to $\mu(G) = 2$ and the only graphs with this value of the arrow number are the ones in which every vertex has even valency.

4. Reduction to trivalent graphs

In this section, it is shown that a graph which is M-arrowable, $M > 2$, has a (possibly not unique) cubic or trivalent (i.e., with three edges at every vertex) counterpart with the same property, from which it can be reproduced by inverse operations. The study of such trivalent graphs is much easier than the general case as can already be seen from the fact, that a subgraph with as only vertices such with even valency must simply be a system of disjoint circuits, which form their own Eulerian circuit in this case. The first three operations are necessary to associate to every M-arrowable graph a simple, homeomorphically irreducible one, i.e. a graph without multiple edges and without valency-2 vertices. These constructions are:

- A As already remarked in the previous section, the square of the matrix Q is simply $M\mathsf{Q}$, see equation (1.6), so that a graph is M-arrowable if and only if its homeomorphically irreducible counterpart has this property. Actually, deletion or insertion of vertices of valency 2 does not change $\alpha(G, M)$.
- B If G contains an r-fold multiple edge with numbers $u_1, \ldots, u_t$ of arrows from vertex 1 to vertex 2 and $u_{t+1}, \ldots u_r$ in the other direction, there are two possibilities. If the net sum S of the arrows is $\neq 0 \bmod M$, it can be replaced by a single edge carrying $S \bmod M > 0$ arrows from 1 to 2 if $S < (M/2)$, by $M - S$ arrows from 2 to 1 if $S > (M/2)$ or by a thick edge if $S = (M/2)$. An example is shown in figure 1(*a*) for the case $M = 5$. Conversely, an edge carrying S arrows can always be replaced by an r-fold multiple one.
- C On the other hand, if this sum is $S = 0 \bmod M$, all these edges can be deleted without changing the M-arrowability of the graph. This is shown in figure 1(*b*) (for cases $M > 4$). In the second step in this figure, construction A has been used twice to obtain a graph without vertices of valency 2. The converse operation simply puts a multiple edge with arrow sum zero (i) between two existing vertices, or (ii) between an existing vertex and a new vertex on an existing edge or (iii) between two such new vertices.

Once this simple graph without vertices of valency 2 is obtained, three further constructions allow the transition to a trivalent graph:

- D Let v be a vertex with valency larger than 3 in a graph G. If the sum of the arrow numbers on three of the edges incident on v is already equal to 0 mod M, then this is also the case for the sum of the arrow numbers on remaining edges, so that a vertex with valency 3 can be dissociated without damaging the arrowing, see figure 1(*c*) for an example valid for

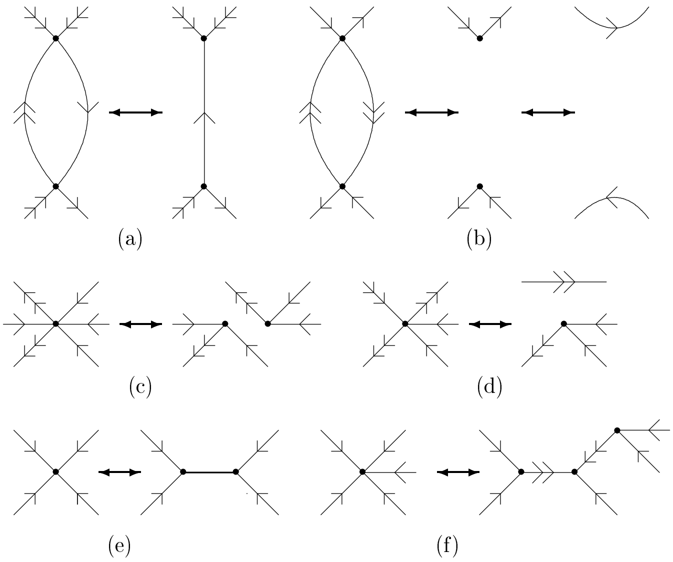

Figure 1. Examples for the constructions B–F to obtain an M-arrowable trivalent graph from an arbitrary one with the same property. See text for descriptions of parts (a)–(f).

all $M > 4$. The converse of this operation is simply the contraction of two nonconnected vertices.

E It is also possible that two of the edges carry equal but opposite numbers of arrows or that there is a pair of thick edges at v. In this case, a vertex of valency 2 can be dissociated, which can subsequently be removed by the construction of A again. It is possible that this operation results in a graph with multiple edges, so that constructions B or C must also be applied to obtain a simple graph. An example, again valid for all $M > 4$, is shown in figure 1(d). The converse of this construction is the insertion of a new vertex on an edge followed by contraction of this vertex with an existing one.

F If, after performing all possible constructions of types D and E, the graph is still not trivalent, an arbitrary pair of edges incident on v can be dissociated and connected to a new vertex, which is in turn connected to v by a new edge. Since no construction of type E was possible, the new edge necessarily carries a nonzero number of arrows. An example of this construction is shown in figure 1(e) for the case $M = 4$, whereas figure 1(f) shows this construction applied twice for an example with $M = 5$. As in case D, the converse procedure is vertex contraction, but between two connected vertices.

Repeated application of the constructions A–F now obviously implies:

(a) For an arbitrary M-arrowable graph G with a given proper arrowing, the constructions yield one or more trivalent, properly arrowed graphs.

(b) Given a properly M-arrowed trivalent graph, a set of M-arrowable graphs can be constructed from this by the inverses of the operations A–F.

These graphs are multiply connected, but may contain vertices of valency 2 as well as multiple edges.

In a properly arrowed trivalent graph, the possible vertex configurations fall into four different classes, due to the fact that there are at most $(M/2)$ arrows on all edges:

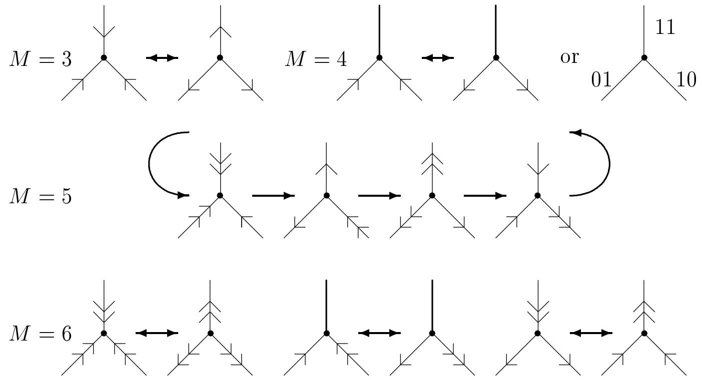

Figure 2. Sinks, sources, bifurcations and confluences for $3 \leqslant M \leqslant 6$.

(1) A *sink* has all three edges pointing towards the vertex; the sum of the numbers of arrows is M.
(2) A *source* is obtained from a sink by inverting all arrows.
(3) At a *bifurcation*, there is one edge pointing toward the vertex carrying u_1 arrows, the other two edges point away from the vertex and their arrow numbers u_2 and u_3 satisfy $u_1 = u_2 + u_3$.
(4) A *confluence* is obtained from a bifurcation by reversing all arrows.

This classification is unique for odd values of M; for even M, the thick edges must be given a fixed, but arbitrary orientation in order to to make it unique. Figure 2 shows the possible configurations for $3 \leqslant M \leqslant 6$. These are the only ones possible by Seymour's theorem [11].

- For $M = 3$ there are only sinks and sources, connected by the automorphism σ, which inverts all elements of an Abelian group [14].
- For $M = 4$, the 4-arrowing based on $\mathcal{C}(4)$ gives the two possible vertices shown, which are again mapped onto each other by σ. The first of these can be interpreted as a sink or as a confluence, the second as a bifurcation or as a source, depending on whether the direction of the thick edge is chosen toward the vertex or away from it, respectively. The 'arrowing' based on $\mathcal{S}(2) \otimes \mathcal{S}(2)$ leads to a single type of vertex, also shown in figure 2. The automorphism group Aut$[\mathcal{S}(2) \otimes \mathcal{S}(2)] \cong \mathcal{S}(3)$ [14, 15] permutes the edge labels in all possible ways.
- For $M = 5$, there is exactly one vertex of the types 1–4 above; these are permuted cyclically by Aut$[\mathcal{C}(5)] \cong \mathcal{C}(4)$ [14, 15].
- For $M = 6$, there are a source–sink and a confluence–bifurcation pair connected by σ again. There is also a pair of vertices with thick edges, which can be interpreted as a sink-bifurcation or a confluence–source pair as in the case $M = 4$.

Another reason to restrict oneself to trivalent graphs comes from a study of the critical graphs $T(M)$, which are minimal and not M-arrowable:

(a) $T(M)$ is not M-arrowable.
(b) If G is such that either $|V(G)| < |V[T(M)]$ holds or that $|V(G)| = |V[T(M)]|$ and $|E(G)| < |E[T(M)]|$ both hold, then G is M-arrowable. The graphs $T(M)$ are all trivalent for $M > 2$, since if they contain a vertex with valency larger than 3, two edges may be dissociated from this vertex and soldered together as in construction E above. The resulting graph has one edge less than $T(M)$, so that it is M-arrowable by part (b)

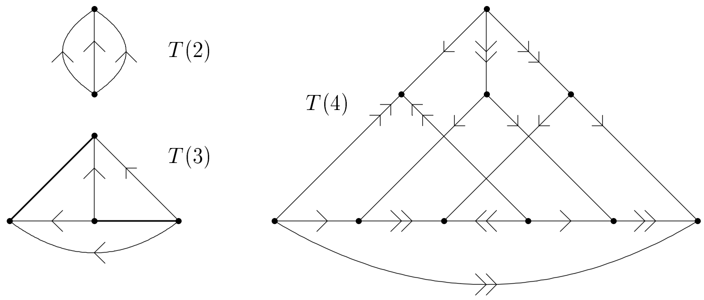

Figure 3. The critical graphs $T(2)$, $T(3)$ and $T(4)$ together with one of their 3-, 4- and 5-arrowings.

of the above definition. (Strictly, this is only true if the resulting graph has no isthmus; that the two edges can be chosen so that this is the case, follows from [16].) Then the converse of construction E would reproduce $T(M)$ as an M-arrowable graph again, which is a contradiction. For $M = 2$, the smallest graph that is not 2-arrowable can also be defined as a trivalent (but nonsimple) one: it simply consists of two vertices connected by three parallel edges, see figure 3 and the next section. It is clear from section 2 that the $T(M)$ are 3-edge-connected and have only trivial 3-cuts. It can be shown that all circuits in $T(M)$ have length at least $2M - 3$.

5. Results for Ising thickness 2 and planar graphs

As shown in figure 2, the only possibilities for the vertices of a properly 3-arrowed trivalent graph are sinks and sources. Since every edge must connect such a sink–source pair, this is only possible if the graph is *bipartite*, i.e. such that its vertices can be coloured with only two colours without neighbouring vertices having the same colour. Conversely, if a trivalent graph is bipartite, it suffices to designate the vertices of one colour as sinks, the others as sources and to put an arrow from every source to every sink:. *A trivalent graph is properly 3-arrowable if and only if it is bipartite.*

The smallest trivalent graph, which is not bipartite, is the complete graph $K(4)$ with four vertices, so that this must be the (unique) critical graph $T(3)$. It is shown in figure 3, where a proper 4-arrowing is also indicated, so that $\mu[T(3)] = 4$ holds. Also shown in figure 3 is the graph $T(2)$ of the previous section, consisting of three parallel edges; this is bipartite, so that $\mu[T(2)] = 3$ follows. A proper 3-arrowing is also shown in figure 3.

Since a graph with $\mu(G) = 3$ has Ising thickness $t(G) = 2$ by equation (3.15), a trivalent bipartite graph must consist of the union of two circuit systems $\mathcal{A}$ and $\mathcal{B}$, a system of disjoint circuits being the only type of subgraph with vertices of even valency only. Now at a vertex of $G = \mathcal{A} \cup \mathcal{B}$, both circuit systems are necesarily present: one edge belongs to $\mathcal{A}$ only, one to $\mathcal{B}$ only and the third one to both circuit systems. Replacing the single arrow on this last edge by a double one in the opposite direction shows that the circuit systems are oriented so that they have a common direction on the shared edges. On the other hand, given a pair of circuit systems which make up G, such a common orientation on the shared edges generates a proper 3-arrowing upon the replacement of a double arrow by a single one in the opposite direction. It is useful to describe this situation by a general definition, valid also for the case that there are other circuit systems necessary to obtain the full graph: two systems of circuits $\mathcal{A}$ and $\mathcal{B}$ are called *compatible*, notation $(\mathcal{A}, \mathcal{B})$ if these circuits can be given an orientation which is the same on all common edges. The same definition applies if the circle systems are, for a general graph, replaced by Eulerian circuit systems.

In the above the following has been proved. *A trivalent graph with Ising thickness $t(G) = 2$ has arrow number $\mu(G) = 3$ if and only if there is a pair of compatible ciruit systems with union G.*

For the case of a general graph with $t(G) = 2$, the inverse constructions of A–F above transform a circuit system into an Eulerian circuit with the same orientation, so that this implies that *a graph with Ising thickness $t(G) = 2$ has arrow number $\mu(G) = 3$ if and only if it is the union of two subgraphs with even-valency vertices throughout, which are such that their Eulerian circuits $\mathcal{A}$ and $\mathcal{B}$ can be oriented so, that they have a common direction on all shared edges.*

A trivalent graph that is not bipartite has $\mu(G) > 3$ necessarily. If it is 4-arrowable, $\mu(G) = 4$ follows. In a 4-arrowing based on $\mathcal{C}(4)$, the only two vertices are the ones shown in figure 2: there is always a thick edge, counting 2, and there are two singly arrowed edged, which either both point towards the vertex or away from it. Tracing a path through these singly arrowed edges in such a fashion that their orientation alternates in the direction of the path must finally give a closed circuit with even circumference. If this visits all vertices, this is a Hamiltonian cycle (see, e.g., [13, p 103]). If not, a similar circuit can be traced out. Finally, one ends up with a system of circuits of even circumference visiting all vertices and with the $|V(G)|$ thick edges as a nonconnected graph. A regular (i.e., with fixed valency k at every vertex) subgraph of a graph containing all vertices is called a k-factor of the graph (see, e.g., [7]), so that the above decomposition is into a 2-factor and a 1-factor. Now such a decomposition is always possible for a trivalent graph without isthmuses (see [13, pp 183–5]). However, the 2-factor found above is special in that all of its circuits have even circumference. It will be called an *even* 2-factor in what follows. On the other hand, it is obvious that if a trivalent graph has an even 2-factor, its edges can be oriented alternatingly; then taking the thick edges as the remaining 1-factor, a proper 4-arrowing results: *a trivalent graph has $\mu(G) = 4$ if and only if it is not bipartite and has an even 2-factor.*

The same result follows from use of the group $\mathcal{S}(2) \otimes \mathcal{S}(2)$; in this case, all vertices have edges labelled by (01), (10) and (11), as shown in figure 3. This can be interpreted as the possibility of colouring the edges of the graph with three colours, so that no equally coloured edges meet at a vertex. Now the edges of a fixed colour form a 1-factor, so that 4-arrowability now decomposes the graph into three 1-factors. However, the union of any two such 1-factors is an even 2-factor necessarily, and the existence of such a 2-factor guarantees the solubility of the edge colouring problem: simply colour the edges of the 2-factor with two alternate colours and use the third one for the remaining 1-factor. This shows explicitly that the arrowing problem does not depend on the group used, as, of course, should be the case. It follows from the above that a 4-arrowable graph can also be written as the union of two even 2-factors, for instance of the 2-factor consisting of (01) and (11) and of the one consisting of the (10) and (11) edges.

The above immediately gives the following result for planar graphs: *a planar trivalent, nonbipartite graph H_p has $\mu(H_p) = 4$.*

Indeed, by the four-colour theorem [18, 19], the faces of a planar graph can be coloured with at most four colours, so that no two faces which have an edge in common have the same colour. These four colours can be labelled by the elements of $\mathcal{S}(2) \otimes \mathcal{S}(2)$; now label the common edge between two faces by the sum (in the group). It is clear that the three faces around a vertex of a trivalent graph all are coloured differently, so that the edges are always labelled by (01), (10) and (11). This is a proper 4-arrowing, so that $\mu(H_p) \leqslant 4$ follows. Since H_p is not bipartite, the result follows.

This now completely solves the problem of the occurring planar graphs in a high-temperature expansion of the Potts model on a two-dimensional lattice graph. The graphs

G_p occurring in the above case are:

- Even-valency graphs occur for all $M \geqslant 2$.
- Graphs with Ising thickness 2, so that their two Eulerian circuit systems are compatible, contribute for all $M \geqslant 3$.
- All other planar graphs contribute for $M \geqslant 4$.

This follows, since the operations A–F and their inverses can be applied so as to leave the planarity invariant (as in figure 1).

Note that the four-colour theorem is really necessary for these proofs to work since $\mu(G_p) = 4$ for a planar graph implies that its faces need at most four colours for a proper colouring. If a planar graph needing five colours existed, it could only contribute for $M \geqslant 5$. It is, therefore, not remarkable that the critical graph $T(4)$ is not planar. This smallest trivalent graph not containing an even 2-factor has long been known in graph theory. It is the Petersen graph [17] with ten vertices. This (again unique) critical graph $T(4)$ is also shown in figure 3 together with a proper 5-arrowing, thus establishing $\mu[T(4)] = 5$. Its Ising thickness is 3 by equation (3.15).

6. Graphs with Ising thickness 3

As remarked above, the Petersen graph has $\mu[T(4)] = 5$, implying that its Ising thickness is $t[T(4)] = 3$. Generally, such trivalent graphs need three circuit systems $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ to cover them. Recall that these circuit systems define a proper 8-arrowing since they define a triple (a, b, c), $a, b, c \in \{0, 1\}$, $(a, b, c) \neq (0, 0, 0)$, for each edge e, where $a = 1$ if $e \in \mathcal{A}$, $b = 1$ if $e \in \mathcal{B}$, $c = 1$ if $e \in \mathcal{C}$. Since $t(G) = 3$ as such only implies $5 \leqslant \mu(G) \leqslant 8$, there must be other conditions on the circle systems to differentiate between the possible values of the arrow number. One possible notion restricting the arrow number, the compatibility of a pair of circuit systems, has been defined in the previous section as the possibility to orient their circuits so that they have a common orientation on all shared edges. This is indeed the essence of Seymour's proof [11]; it is shown that *all* trivalent graphs without an isthmus must be a union of two compatible circuit systems $(\mathcal{A}, \mathcal{B})$, which constitute a 3-arrowable subgraph, and a third circuit system $\mathcal{C}$. Then $\mu(G) \leqslant 6$ follows from equation (3.14).

The question as to whether this notion is also satisfactory in distinguishing the cases $\mu(G) = 5$ and is still open. The following results hold:

(a) *The three circuit systems $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are not all mutually compatible.* If this were the case, then it suffices to put one arrow on these, so that there two parallel arrows on edges common to two circuit systems and three on the edges belonging to all. This would imply $\mu(G) \leqslant 4$, which is a contradiction for $t(G) = 3$.

(b) *If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{C})$ both hold, then $\mu(G) = 5$.* Indeed, let the edges of $\mathcal{A}$ and $\mathcal{C}$ be given 1 and 2 arrows, respectively; $\mathcal{B}$ is compatible with both, so that it can be oriented parallel to $\mathcal{A}$ and antiparallel to $\mathcal{C}$. Therefore, it can be given 3 arrows in such a way, that on edges common to $\mathcal{A}$ and $\mathcal{C}$ there are 4 arrows, on edges common to $\mathcal{B}$ and $\mathcal{C}$ there is one arrow and on edges common to all there are only 2 arrows. This implies that all edges carry at most 4 arrows, so that $\mu(G) = 5$ since $t(G) = 3$. Presumably, a proof of Tutte's conjecture [6] would show that all trivalent graphs with $t(G) = 3$ have circuit systems of this type. For the Petersen graph, it can easily be shown to be the case.

Now we are in a position to characterize all graphs occurring in the high-temperature of the Potts model in three or more dimensions:

- All graphs with even-valency vertices throughout ($t(G) = 1$) contribute for all $M \geqslant 2$.

- Graphs with $t(G) = 2$ and compatible circuit systems (trivalent graphs are bipartite) contribute for all $M \geqslant 3$.
- All other graphs with $t(G) = 2$ (trivalent ones contain an even 2-factor) are present for all $M \geqslant 4$.
- For $t(G) = 3$, the graphs with $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{C})$ (a trivalent example is the Petersen graph) contribute for all $M \geqslant 5$.
- All other graphs with $t(G) = 3$, which comprise all finite graphs not listed already, contribute for all $M \geqslant 6$.

The set of graphs defined in the last item is empty if Tutte's conjecture holds.

The above completely solves the problem stated in the introduction in a formal way. The practical question of finding the arrow number for a given graph has not been touched upon in this paper. A brute-force search through all possible arrowings to find a proper one is obviously out of the question. For certain classes of graphs, an answer can sometimes be found by other means. Two examples will be sketched here: the first one concerning a generalization of $T(3)$, the second one of $T(4)$.

(i) By decomposing a graph into two edge-disjoint ones, the implication

$$G = G_1 \cup G_2 \qquad E(G_1) \cap E(G_2) = \emptyset \Rightarrow \mu(G) \leqslant \max[\mu(G_1), \mu(G_2)] \tag{6.1}$$

is clear. The complete graphs $K(2q)$ can, for $q > 2$, be decomposed into a trivalent, bipartite graph and one with even valency, so that $\mu[K(2q)] = 3$ follows.

(ii) Generalized Petersen graphs $P(m, k)$ [20] are graphs consisting of two circuits of circumference m, the vertices of which are connected so, that the ith one of the first circuit ($i = 0, 1, \ldots, m-1$) is connected to the one with number ki on the second circuit (k and m have no factors in common). The Petersen graph is $P(5, 2)$ in this notation. Nearly all these generalized graphs can be shown to have $\mu(G) = 4$ in contrast to $P(5, 2)$ by treating the values of a proper 4-arrowing on the connecting edges as a formal (regular) language and showing that the circuits can 'understand' each other for $M = 4$. For a subclass of these graphs, the result also follows from the known presence of a Hamiltonian cycle, which forms an even factor [20].

It is interesting that for both $T(3)$ and $T(4)$, their generalizations lead to graphs with smaller arrow number. Possibly such methods can also be applied to other classes of graphs.

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